Lecture 27. Complex eigenvalues

<u>Def</u> A <u>complex number</u> is a number of the form $a+bi$ with $a, b \in \mathbb{R}$

where i denotes the imaginary square root of -1.

<u>Note</u> A square matrix may have nonreal eigenvalues

e.g. $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} \Longrightarrow p_A(\lambda) = \lambda^2 - (2+2)\lambda + (2\cdot 2 - 3\cdot(-3)) = \lambda^2 - 4\lambda + 13$

$\Longrightarrow$ A has eigenvalues $\lambda = 2 \pm 3i$

<u>Prop</u> Every complex vector $\vec{v}$ can be uniquely written as

$$\vec{v} = \mathrm{Re}(\vec{v}) + i\,\mathrm{Im}(\vec{v})$$

where $\mathrm{Re}(\vec{v})$ and $\mathrm{Im}(\vec{v})$ are real vectors.

<u>Note</u> (1) $\mathrm{Re}(\vec{v})$ and $\mathrm{Im}(\vec{v})$ are respectively called the <u>real part</u> and the <u>imaginary part</u> of $\vec{v}$.

(2) We often consider in pair with $\overline{\vec{v}} = \mathrm{Re}(\vec{v}) - i\,\mathrm{Im}(\vec{v})$, called the <u>conjugate</u> of $\vec{v}$.

e.g. $$\underbrace{\begin{bmatrix} 3+4i \\ 1-6i \\ 2+i \end{bmatrix}}_{\vec{v}} = \underbrace{\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}}_{\mathrm{Re}(\vec{v})} + i \underbrace{\begin{bmatrix} 4 \\ -6 \\ 1 \end{bmatrix}}_{\mathrm{Im}(\vec{v})} \Longrightarrow \overline{\begin{bmatrix} 3+4i \\ 1-6i \\ 2+i \end{bmatrix}} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} - i \begin{bmatrix} 4 \\ -6 \\ 1 \end{bmatrix} = \begin{bmatrix} 3-4i \\ 1+6i \\ 2-i \end{bmatrix}$$

<u>Thm</u> Let A be a square matrix whose entries are real numbers.

If $\vec{v}$ is a complex eigenvector of A with eigenvalue λ, then

$\overline{\vec{v}}$ is a complex eigenvector of A with eigenvalue λ

Thm If a 2×2 matrix A has a complex eigenvector $\vec{v}$ with nonreal eigenvalue $\lambda = a - bi$, we have

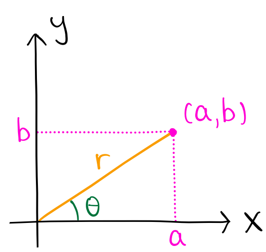
$$A = PCP^{-1}$$

with $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ and P having $\mathrm{Re}(\vec{v})$, $\mathrm{Im}(\vec{v})$ as columns

Note (1) C is called a scaled rotation matrix, since it can be written as a multiple of a rotation matrix. In fact, we have

$$C = r\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

where $r = \sqrt{a^2+b^2}$ and θ respectively denote the length and angle components for (a, b) in polar coordinates



e.g. $C = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \Longrightarrow C = 2\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = 2\begin{bmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{bmatrix}$

(2) The negative sign for the imaginary part of $\lambda = a - bi$ is to ensure that the rotation angle for C goes counterclockwise.

(3) Powers of A are described as follows:

$A = PCP^{-1} \Longrightarrow A^n = PC^nP^{-1}$ (cf. Lecture 25)

$$C = r\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \Longrightarrow C^n = r^n \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^n}_{\text{rotation by } \theta \text{ iterated } n \text{ times}} = r^n \underbrace{\begin{bmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{bmatrix}}_{\text{rotation by } n\theta}$$

Ex Consider the matrix

$$A = \begin{bmatrix} 3 & -1 \\ 5 & -1 \end{bmatrix}$$

(1) Find all (complex) eigenvalues of A

Sol $p_A(\lambda) = \lambda^2 - (3-1)\lambda + (3\cdot(-1) - 5\cdot(-1)) = \lambda^2 - 2\lambda + 2$

$\Longrightarrow$ A has eigenvalues $\boxed{\lambda = 1 \pm i}$

(2) Find a complex eigenvector for each eigenvalue of A.

Sol Since A is a 2×2 matrix, it is advisable to solve the equation

$A\vec{x} = \lambda\vec{x}$ without computing $\mathrm{RREF}(A - \lambda I)$.

For $\lambda = 1+i$, we have

$$\begin{cases} 3x_1 - x_2 = (1+i)x_1 \\ 5x_1 - x_2 = (1+i)x_2 \end{cases}$$

We can find a solution by only considering the first equation.

($\mathrm{RREF}(A-\lambda I) \neq I \Longrightarrow \mathrm{RREF}(A-\lambda I)$ has only one nonzero row

$\Longrightarrow$ The two equations are equivalent)

$\Longrightarrow (2-i)x_1 = x_2$

We set $x_1 = 1$ to obtain a solution $\vec{x} = \begin{bmatrix} 1 \\ 2-i \end{bmatrix}$

For $\lambda = 1+i$, we take the conjugate $\overline{\vec{x}} = \begin{bmatrix} 1 \\ 2+i \end{bmatrix}$

Hence A has complex eigenvectors

$$\boxed{\begin{bmatrix} 1 \\ 2-i \end{bmatrix} \text{ for } \lambda = 1+i} \text{ and } \boxed{\begin{bmatrix} 1 \\ 2+i \end{bmatrix} \text{ for } \lambda = 1-i}$$

(3) Find an invertible matrix P and a scaled rotation matrix with $A = PCP^{-1}$.

Sol The eigenvector $\vec{v} = \begin{bmatrix} 1 \\ 2+i \end{bmatrix}$ with eigenvalue $\lambda = 1-i$ yields

$$\boxed{P = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}$$

$\operatorname{Re}(\vec{v})$ $\operatorname{Im}(\vec{v})$

(4) Compute A^{20}.

Sol $C = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \sqrt{2}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \sqrt{2}\begin{bmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{bmatrix}$

($r = \sqrt{2}$ and $\theta = \pi/4$ for $(1,1)$)

$$\Rightarrow C^{20} = (\sqrt{2})^{20}\begin{bmatrix} \cos(20\cdot\pi/4) & -\sin(20\cdot\pi/4) \\ \sin(20\cdot\pi/4) & \cos(20\cdot\pi/4) \end{bmatrix}$$

$$= 2^{10}\begin{bmatrix} \cos(5\pi) & -\sin(5\pi) \\ \sin(5\pi) & \cos(5\pi) \end{bmatrix}$$

$$= 2^{10}\begin{bmatrix} \cos(\pi) & -\sin(\pi) \\ \sin(\pi) & \cos(\pi) \end{bmatrix}$$

(sine and cosine have period 2π)

$$= 2^{10}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = -2^{10} I$$

$$\Rightarrow A^{20} = PC^{20}P^{-1} = P(-2^{10}I)P^{-1} = -2^{10}\cancel{PIP^{-1}} = -2^{10}I$$

$$= \boxed{\begin{bmatrix} -2^{10} & 0 \\ 0 & -2^{10} \end{bmatrix}}$$